On the Dynamics of Bianchi IX Cosmological Models

Hossein Farajollahi · Arvin Ravanpak

Abstract A cosmological description of the universe is proposed in the context of Hamiltonian formulation of a Bianchi IX cosmology minimally coupled to a massless scalar field. The classical and quantum results are studied with special attention to the case of closed Friedmann-Robertson-Walker model.

Keywords General relativity · Quantum cosmology · Time · Bianchi IX · FRW

1 Introduction

Any theoretical scheme of gravity must address a variety of conceptual issues including the problem of time and identification of dynamical observables [1–10]. Studying cosmological models instead of general relativity helps us to overcome the problems related to the infinite number of degrees of freedom in the theory and pay more attention to the issues arising from the time reparametrization invariance of the theory; such as the identification of a dynamical time and also construction of observables for the theory [11–14]. In particular, Bianchi IX cosmological model is an interesting candidate to test the possible solutions of the above problems [15–17].

The problem of time is the most known difficulty of the Wheeler-DeWitt (WDW) quantum geometrodynamics which is a theoretical basis for modern quantum cosmology [18, 19]. Time in quantum mechanics and general relativity are drastically different from each other; in quantum mechanics time is a global and absolute parameter, in general relativity a local and dynamical variable [20]. As a result, the wave function in quantum formulation is time independent, i.e., the universe has a static picture.

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A common candidate for a dynamical time in classical and quantum model is a massless scalar field coupled to gravity [14, 15, 21]. In this work we apply this to Bianchi IX cosmological model and investigate its classical and quantum implementation. Due to the presence of the minimally coupled scalar field term in the formulation, we show that the dynamics of the metric functions can be obtained using a time variable, as a function of the scalar field.

The article is organized as follows: In Sect. 2 the Hamiltonian formulation of Bianchi IX model coupled to a massless scalar field is studied, with particular attention to the closed FRW model. In Sect. 3 the canonical quantization of the model is presented and the wave function of the universe is discussed. Again special attention is given to the quantum FRW cosmological solutions with closed curvature. Section 4 presents conclusion drawn from this work.

2 The Classical Model

The characteristic feature of the Bianchi IX model is the existence of the simply transitive isometry group. The infinitesimal generators of this group are the three linearly independent spacelike killing vectors, ξ_i, which obey

$$[\xi_i, \xi_j] = C_{ij}{}^k \xi_k, \tag{1}$$

where i, j and k runs from 1 to 3. The generators of these transformations are

$$\begin{aligned}
\xi_1 &= x^2 \partial_3 - x^3 \partial_2, \\
\xi_2 &= x^3 \partial_1 - x^1 \partial_3, \\
\xi_3 &= x^1 \partial_2 - x^2 \partial_1.
\end{aligned} \tag{2}$$

One may write the metric of Bianchi IX in terms of *forms*,

$$ds^2 = N^2 dt^2 - h_{ij} \sigma^i \sigma^j, \tag{3}$$

where N is the lapse function, h_{ij} is the 3-metric and the σ_i are the left invariant *1-forms* of $SU(2)$:

$$\begin{aligned}
\sigma^1 &= \cos\psi \, d\theta + \sin\psi \sin\theta \, d\phi, \\
\sigma^2 &= -\sin\psi \, d\theta + \cos\psi \sin\theta \, d\phi, \\
\sigma^3 &= d\psi + \cos\theta \, d\phi,
\end{aligned} \tag{4}$$

with $0 \leq \psi \leq 4\pi$, $0 \leq \theta \leq \pi$ and $0 \leq \phi \leq 2\pi$.

Without loss of generality, we can assume that the metric h_{ij} is diagonal. Using the Misner variables to parameterize the metric

$$h_{ij} = e^{2\alpha} (e^{2\beta})_{ij}, \tag{5}$$

where α and β are functions of time only. The matrix

$$\beta_{ij} = \text{diag}[\beta_+ + \sqrt{3}\beta_-, \beta_+ - \sqrt{3}\beta_-, -2\beta_+], \tag{6}$$

with the property $\text{Tr}\beta = 0$, ensures that the 3-volume of the hypersurface depends only on the conformal factor α. The variables β_+ and β_- describe the anisotropy of the spacetime, and, in particular, if they equal to zero, the model reduces to the ordinary closed FRW model.

The Hilbert action for the model, including the massless scalar field minimally coupled to gravity, is given by

$$S_H = \int \sqrt{-g}\left(R - \frac{1}{2}g^{\mu\nu}\phi_{,\mu}\phi_{,\nu}\right)d^4x, \tag{7}$$

where μ and ν runs from 1 to 4 and $\sqrt{-g} = N\sqrt{h}$. If one assume that the scalar field is spatially homogeneous, the Lagrangian for the model in terms of the Misner variables becomes

$$L = \frac{-6e^{3\alpha}}{N}(\dot{\alpha}^2 - \dot{\beta}_+^2 - \dot{\beta}_-^2) + \frac{Ne^\alpha}{2}V(\beta_+, \beta_-) - \frac{e^{3\alpha}}{2N}\dot{\phi}^2, \tag{8}$$

where

$$V(\beta_+, \beta_-) = e^{-8\beta_+} - 4e^{-2\beta_+}\cosh(2\sqrt{3}\beta_-) + 2e^{4\beta_+}(\cosh(4\sqrt{3}\beta_-) - 1). \tag{9}$$

For the Hamiltonian formulation of the theory, let's determine the conjugate momenta to the dynamical variables α, β_+, β_-, ϕ, and N:

$$P_\alpha = \frac{\partial L}{\partial\dot{\alpha}} = \frac{-12e^{3\alpha}\dot{\alpha}}{N}, \tag{10}$$

$$P_+ = \frac{\partial L}{\partial\dot{\beta}_+} = \frac{12e^{3\alpha}\dot{\beta}_+}{N}, \tag{11}$$

$$P_- = \frac{\partial L}{\partial\dot{\beta}_-} = \frac{12e^{3\alpha}\dot{\beta}_-}{N}, \tag{12}$$

$$P_\phi = \frac{\partial L}{\partial\dot{\phi}} = -\frac{e^{3\alpha}\dot{\phi}}{N}, \tag{13}$$

$$P_N = \frac{\partial L}{\partial\dot{N}} = 0. \tag{14}$$

Obviously, the variable N is not a canonical variable as its canonical conjugate momentum is zero. The canonical Hamiltonian can then be written as

$$H_c = \frac{e^{-3\alpha}N}{24}(-P_\alpha^2 + P_+^2 + P_-^2) - \frac{Ne^\alpha V(\beta_+, \beta_-)}{2} - \frac{Ne^{-3\alpha}P_\phi^2}{2}, \tag{15}$$

with the total Hamiltonian to be

$$H_t = H_c + \lambda P_N. \tag{16}$$

Since $P_N \approx 0$ is a primary constraint, preservation of this constraint over time yields a secondary constraint (called Hamiltonian constraint),

$$\mathcal{H} = \frac{e^{-3\alpha}}{24}[P_\alpha^2 - P_+^2 - P_-^2 + 12e^{4\alpha}V(\beta_+, \beta_-)] + \frac{e^{-3\alpha}P_\phi^2}{2} \approx 0. \tag{17}$$

By choosing $N_1 = Ne^{-3\alpha}$ and introducing a new variable $a = e^{\alpha}$, (a is scale factor) we can redefine the secondary constraint as

$$\mathcal{H}_1 = \frac{1}{24}[a^2 P_a^2 - P_+^2 - P_-^2 + 12a^4 V(\beta_+, \beta_-)] + \frac{P_\phi^2}{2} \approx 0, \tag{18}$$

where $P_a = \frac{\partial L}{\partial \dot{a}}$. Thus

$$H_t = \lambda P_N + N_1 \mathcal{H}_1, \tag{19}$$

in which according to Dirac procedure, both of these constraints are first class.

Now, we define a new time variable and its conjugate momentum as [11]:

$$T = \int_{\Sigma_t} \frac{\phi}{P_\phi} \sqrt{h} d^3 x, \tag{20}$$

$$\Pi_T = \frac{P_\phi^2}{2}, \tag{21}$$

where the integration is over the spatial hypersurface Σ_t. The Hamiltonian constraint in terms of the new variables becomes

$$\mathcal{H}_1 = \mathcal{H}^* + \Pi_T \approx 0, \tag{22}$$

where $\mathcal{H}^*$ is the Hamiltonian constraint without scalar field.

The equation of motion for T,

$$\frac{dT}{dt} = \int_{\Sigma_t} N_1 \sqrt{h} d^3 x, \tag{23}$$

shows that the time variable equals the four-volume enclosed between the initial and final hypersurfaces, which is necessarily positive and monotonically increasing and can play the role of a cosmological time. Even though, the time variable is not a Dirac observable, it can be used to play the role of a cosmological time. Besides, the conjugate momentum to this variable is a Dirac observable.

Classically, We can analyze in which sense the dynamical time, T, can be regarded as an appropriate time variable for the theory. Near the cosmological singularity ($a \to 0$) the potential term $V(\beta_+, \beta_-)$ can be neglected and so we have

$$\mathcal{H}_1 = \frac{1}{24}[a^2 P_a^2 - P_+^2 - P_-^2] + \Pi_T \approx 0. \tag{24}$$

In such approximation we obtain the new equations of motion

$$a' = \frac{1}{12} a^2 P_a, \tag{25}$$

$$P_a' = -\frac{1}{12} a P_a^2, \tag{26}$$

$$P_+' = P_-' = 0, \tag{27}$$

$$\beta_+' = -\frac{1}{12} P_+, \tag{28}$$

$$\beta_-' = -\frac{1}{12} P_-, \tag{29}$$

which , $(...)'$, denotes derivative of $(...)$, with respect to T. The solutions to the above system for the scale factor a and its conjugate momenta are

$$a = a_0 e^{CT}, \tag{30}$$

$$P_a = P_0 e^{-CT}, \tag{31}$$

where $a_0 = a|_{T=0}$, $P_0 = P|_{T=0}$ and C is a constant. A positive C with positive a_0 provides an accelerating expanded universe, with a positive constant Hubble, $H = C$. We have recovered a monotonic dependence of our new variable T with respect to the isotropic variable of the Universe a and therefore T shows to be a relational time variable for the gravitational dynamics. In this case, the universe expands exponentially according to the cosmological time variable, and naturally accelerating without a beginning singularity.

In the presence of the potential term, i.e., far from the singularity, the equations of motion are

$$a' = \frac{1}{12} a^2 P_a, \tag{32}$$

$$P_a' = -\frac{1}{12} a P_a^2 - 2a^3 [e^{-8\beta_+} - 4e^{-2\beta_+} \cosh(2\sqrt{3}\beta_-)$$
$$+ 2e^{4\beta_+} (\cosh(4\sqrt{3}\beta_-) - 1)], \tag{33}$$

$$\beta_+' = -\frac{1}{12} P_+, \tag{34}$$

$$\beta_-' = -\frac{1}{12} P_-, \tag{35}$$

$$P_+' = -4a^4 [-e^{-8\beta_+} + e^{-2\beta_+} \cosh(2\sqrt{3}\beta_-)$$
$$+ e^{4\beta_+} (\cosh(4\sqrt{3}\beta_-) - 1)], \tag{36}$$

$$P_-' = -4\sqrt{3} a^4 [-e^{-2\beta_+} \sinh(2\sqrt{3}\beta_-) + e^{4\beta_+} \sinh(4\sqrt{3}\beta_-)]. \tag{37}$$

Although, it is hard to solve these set of coupled nonlinear differential equations analytically, a numerical solution of the equations shows, in particular, the dynamic of the scale factor in terms of the time variable T (Fig. 1). Again, We have recovered a monotonic dependence of our new variable T with respect to the isotropic variable of the Universe a and the universe expansion begins with no singularity.
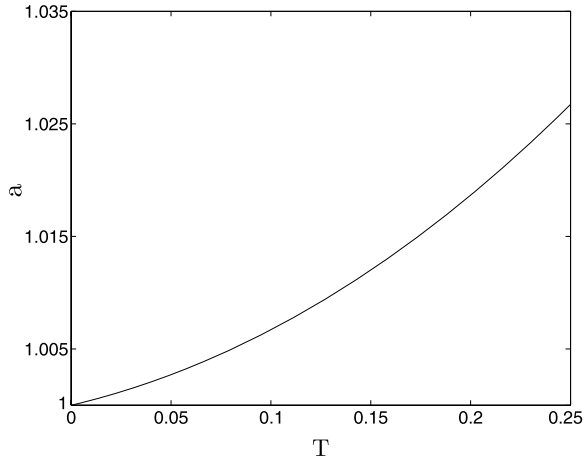
2.1 Isotropic Bianchi Type IX

In particular, we are more interested to find the behavior of the isotropic variables a and p_a when $\beta_+ = 0$ and $\beta_- = 0$. We find the equations of motions as,

$$a' = \frac{a^2 P_a}{12}, \tag{38}$$

and

$$P_a' = -\frac{a P_a^2}{12} - 6a^3. \tag{39}$$

Fig. 1 Dynamics of scale factor with respect to T

The nontrivial solution of the above coupled nonlinear differential equations (38) and (39), for the scale factor gives

$$a(T) = \frac{2\sqrt{c_1}e^{\sqrt{\frac{c_1}{2}}(T+c_2)}}{(4c_1 + e^{2\sqrt{c_1}(T+c_2)})^{\frac{1}{2}}}, \tag{40}$$

where c_1 and c_2 are positive constants of integrations. This solution shows that the universe has no singularity at all. Besides, the universe shrinks to a big crunch as T goes to infinity while it reaches a maximum size during its history. Note that, in the isotropic limit, the Bianchi IX model contains the closed FRW model and the behavior of the scale factor a is as expected for this model.

3 The Quantum Model

In quantization of the theory, we obtain the Wheeler-DeWitt equation

$$\widehat{\mathcal{H}}_1\psi = 0, \tag{41}$$

or

$$\widehat{\mathcal{H}^*}\psi = -\Pi_T\psi. \tag{42}$$

So, we get a Schrödinger like equation

$$\widehat{\mathcal{H}^*}\psi = i\hbar\frac{\partial}{\partial T}\psi, \tag{43}$$

where the new variable plays the role of a dynamical time for the theory. Explicitly, the Schrödinger equation can be written as:

$$\frac{1}{24}\left[a^2\frac{\partial^2}{\partial a^2} - \frac{\partial^2}{\partial \beta_+^2} - \frac{\partial^2}{\partial \beta_-^2}\right]\psi = i\hbar\frac{\partial}{\partial T}\psi, \tag{44}$$

where we have ignored the potential term in this regime.

In fact, in quantum cosmology, the Universe is described by a single wave function ψ providing puzzling interpretations when analyzing the differences between ordinary quantum mechanics and quantum cosmology. If we consider the universe wave function as:

$$\psi(a, \beta_+, \beta_-; T) = \mathcal{A}(a)B_+(\beta_+)B_-(\beta_-)\mathcal{T}(T), \tag{45}$$

then, by using separation of variables method we obtain

$$i\hbar \frac{1}{\mathcal{T}}\frac{d\mathcal{T}}{dT} = E, \tag{46}$$

$$\frac{1}{B_+}\frac{d^2 B_+}{d\beta_+^2} + m^2 = 0, \tag{47}$$

$$\frac{1}{B_-}\frac{d^2 B_-}{d\beta_-^2} + n^2 = 0, \tag{48}$$

$$\frac{a^2}{\mathcal{A}}\frac{d^2 \mathcal{A}}{da^2} + l^2 = 0, \tag{49}$$

where $l^2 = m^2 + n^2 + k^2$, $k^2 = \frac{24\pi E}{\hbar^2}$ and E, k, m, n and l are arbitrary constants. The solutions to these equations are:

$$\mathcal{T}(T) = \exp\left(\frac{-iET}{\hbar}\right), \tag{50}$$

$$B_+(\beta_+) = c_1 \cos(m\beta_+) + c_2 \sin(m\beta_+), \tag{51}$$

$$B_-(\beta_-) = c_1' \cos(n\beta_-) + c_2' \sin(n\beta_-), \tag{52}$$

$$\mathcal{A}(a) = \sqrt{a}\left(c_1'' \cos\left(\frac{\sqrt{(1-4l^2)}\ln a}{2}\right) + c_2'' \sin\left(\frac{\sqrt{(1-4l^2)}\ln a}{2}\right)\right), \tag{53}$$

where c_1, c_2, c_1', c_2', c_1'' and c_2'' are normalization coefficients.

For the Hamiltonian operator to be self adjoint the wave function must satisfy one of the following boundary conditions [22, 23]

$$\psi(a, \beta_+, \beta_-, T)|_{a=0} = 0, \tag{54}$$

$$\frac{\partial \psi(a, \beta_+, \beta_-, T)}{\partial a}\bigg|_{a=0} = 0. \tag{55}$$

The first boundary condition is satisfied while the second one leads to infinity. Obviously, the wave function is not square integrable and in order to obtain a possible physical solution we may construct wave packets as [23]

$$\psi(a, \beta_+, \beta_-; T) = \int A(E)\psi_E(a, \beta_+, \beta_-; T)dE. \tag{56}$$

However, the wave packets are also problematic and a satisfactory solution to the WDW equation is not easy to obtain.

3.1 Isotropic Bianchi Type IX

In the case of isotropic universe, the associated Wheeler-DeWitt equation, i.e., the Schrödinger-like equation is

$$-\partial_a^2 \Psi - 36a^2 \Psi - i\frac{24}{a^2}\partial_T \Psi = 0. \tag{57}$$

The solution is

$$\Psi_E(a, T) = e^{-iET}\sqrt{a}[c_3 J_{+\nu}(3a^2) + c_4 J_{-\nu}(3a^2)], \tag{58}$$

where c_3 and c_4 are constants of integrations, $J_{+\nu}$ and $J_{-\nu}$ are Bessel functions and $\nu = \sqrt{1-96E}/4$. Since $J_{-\nu}$ grows exponentially to infinity as a goes to zero, one must set $c_4 = 0$, and consequently the first boundary condition, (54), is satisfied. In this isotropic case the wave function is still not square integrable and even using wave packets does not give us a satisfactory wave function for the universe.

4 Conclusion

In this work, we present the Hamiltonian formulation of the Bianchi type IX cosmological model minimally coupled to a scalar field. We show that the dynamics of the metric functions can be obtained using a time variable, $T(t)$, as a function of the scalar field.

We then apply the results for the classical and quantum models. We find that the classical model has solutions which avoid the usual initial cosmological singularity. Particularly, in a positive curvature spacetime, the solution also shows a big crunch in future. In the quantum description of the model, however, a satisfactory wave function is difficult to be physically interpreted. Even in the case of isotropic Bianchi type IX model, it is not easy to find a normalizable solution to the WDW equation.

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